Dense and closed subsets of compact-like topological spaces

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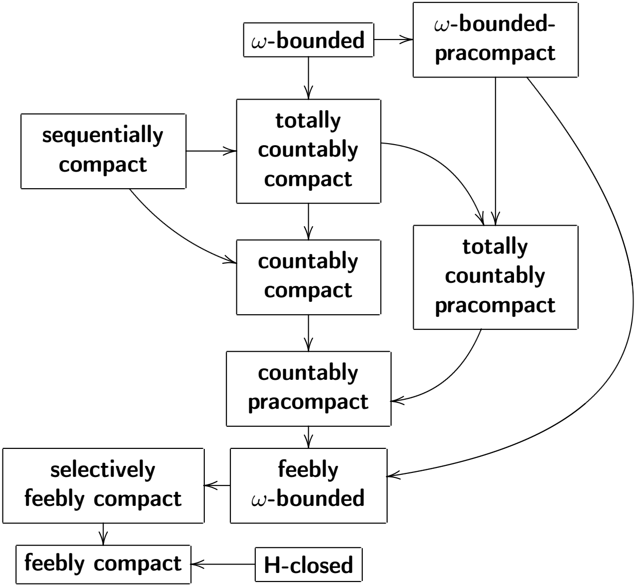
Definition

A topological space X is said to be

- *H-closed* if each open filter on X has an accumulation point;
- *ω-bounded* if each countable subset of X has a compact closure;
- *totally countably compact* if each countable subset of X contains an infinite subset with compact closure;
- *countably compact* if each countable subset of X has an accumulation point;
- *ω-bounded-pracompact*, if there exists a dense subset D in X such that each countable subset of the set D has compact closure in X;
- *totally countably pracompact*, if there exists a dense subset D in X such that each sequence of points of the set D has a subsequence with compact closure in X;
- *countably pracompact* if there exists a dense subset D in X such that every countable subset $A \subseteq D$ has an accumulation point x in X;
- *feebly ω-bounded* if for each sequence $\{U_n\}_{n\in\omega}$ of open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \varnothing$ for each $n \in \omega$;
- *selectively feebly compact*, if for each sequence $\{U_n\}_{n\in\omega}$ of open subsets of X we can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \omega$ such that x is an accumulation point of the set $\{x_n\}_{n\in\omega}$;
- *feebly compact* if each countable open filter on X has an accumulation point.

Diagram



ω-bounded
ω-bounded-pracompact
sequentially compact
totally countably compact
countably compact
totally countably pracompact
countably pracompact
selectively feebly compact
feebly ω-bounded
feebly compact
H-closed

Definition

A semigroup S is called *inverse semigroup* if for each $x \in S$ there exists a unique $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$.

Definition

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively).

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A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$.

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A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$.

Theorem (Andersen, 1952).

A simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup.

Theorem (Eberhart, Selden, 1969).

Topological bicyclic monoid is discrete.

Theorem (Anderson, Hunter, Koch, 1965).

Compact topological semigroups can not contain the bicyclic monoid.

Theorem (Hildebrant, Koch, 1988).

Γ-compact topological semigroups can not contain the bicyclic monoid.

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Theorem (Gutik, Repovš, 2007).

Countably compact topological inverse semigroups can not contain the bicyclic monoid.

Theorem (Banakh, Dimitrova, Gutik, 2010).

- Topological semigroups with countably compact square can not contain the bicyclic monoid;
- topological semigroups with pseudocompact square can not contain the bicyclic monoid.

Theorem (Banakh, Dimitrova, Gutik, 2010).

There exists a pseudocompact topological semigroup which contains densely the bicyclic monoid.

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Problem (Banakh, Dimitrova, Gutik, 2010).

Does there exist a pseudocompact topological semigroup which contains the bicyclic monoid as a **closed** subsemigroup?

Problem 2 (Banakh, Dimitrova, Gutik, 2010).

Does there exist **in ZFC** a countably compact topological semigroup which contains densely the bicyclic monoid?

Problems

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Theorem (2018)

(1) If X is a dense subsemigroup of an ω-bounded-pracompact topological (semitopological, topological inverse, resp.) semigroup then there exists an ω-bounded-pracompact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

(2) If X is a dense subsemigroup of a totally countably pracompact topological (semitopological, topological inverse, resp.) semigroup then there exists a totally countably pracompact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

(3) If X is a dense subsemigroup of a countably pracompact topological (semitopological, topological inverse, resp.) semigroup then there exists a countably pracompact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

(4) If X is a dense subsemigroup of a feebly ω-bounded topological (semitopological, topological inverse, resp.) semigroup then there exists a feebly ω-bounded topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

Second portion of main result

Theorem (2018)

(5) If X is a dense subsemigroup of a selectively feebly compact topological (semitopological, topological inverse, resp.) semigroup then there exists a selectively feebly compact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

(6) If X is a dense subsemigroup of a feebly compact topological (semitopological, topological inverse, resp.) semigroup then there exists a feebly compact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup;

(7) If X is a dense subsemigroup of a pseudocompact topological (semitopological, topological inverse, resp.) semigroup then there exists a pseudocompact topological (semitopological, topological inverse, resp.) semigroup which contains X as a closed subsemigroup.

Nice small result

Theorem (Katetov, 1940)

A H-closed topological space X is compact iff all closed subsets of X are H-closed.

Qustion

Can we describe topological spaces which are closed subsets of H-closed topological spaces?

Proposition (2018)

Each topological space can be embedded as a closed subset into a H-closed topological space.

Proposition (2018)

A discrete semigroup of all finite partial injections of ω can not be embedded into a topological semigroup which is a H-closed topological space.

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Thank You for attention!